Analytical solution for the modified nonlinear Schrödinger equation describing optical shock formation

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We present an exact analytical solution by the use of an ansatz method for the modified nonlinear Schrödinger equation $iU_\zeta+\frac{1}{2}\sigma U_{\tau\tau}+N^2|U|^2U+isN^2(|U|^2U)_\tau=0$, describing the propagation of light pulses in optical fibers. The inclusion of the term $isN^2(|U|^2U)_\tau$ in the usual nonlinear Schrödinger equation arises from an intensity-dependent group velocity and produces a temporal pulse distortion leading to the development of an optical shock. Previous work [Xu Bingzhen and Wang Wenzheng, Phys. Rev. E **51**, 1493 (1995)] using the traveling-wave method does not exhibit this important physical picture. [S1063-651X(98)05004-1]

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I. INTRODUCTION

A fascinating manifestation of fiber nonlinearity occurs in the anomalous-dispersion regime where the fiber can support optical solitons through an interplay between the dispersive and nonlinear effects. The *soliton* refers to special kinds of waves that can propagate undistorted over long distances and remain unaffected after collision with each other. In the context of optical fibers, solitons are not only of fundamental interest but also have potential application in the field of optical fiber communications.

The nonlinear Schrödinger equation (NLSE) has been employed to explain a variety of effects in propagation of pulses in optical fibers, although it only includes self-phase modulation (SPM) and group velocity dispersion (GVD) [1]. However, in other cases a generalized NLSE has been required to account for observations not explained by the NLSE. The generalized NLSE includes high-order nonlinear and dispersive terms. In some particular cases, one can include only one additional term in the NLSE. In the case of optical fibers, when the first derivative of the slowly varying part of nonlinear polarization is added, it leads to a self-steepening of the pulse edge in the absence of dispersion and this modified NLSE (MNLSE) still explains or predicts new nonlinear phenomena [12].

The propagation of a temporal optical soliton in the presence of the self-steepening term can be described by the MNLSE [2]

$$iU_\zeta+\tfrac{1}{2}\sigma U_{\tau\tau}+N^2|U|^2U+isN^2(|U|^2U)_\tau=0, \tag{1}$$

where $U(\zeta,\tau)$ represents a normalized complex amplitude of the pulse envelope, ζ is a normalized distance along the fiber, τ is the normalized time within the frame of the reference moving along the fiber at the group velocity, $\sigma=\pm1$ for the normal and anomalous regime, respectively, the physical significance of N is that its integer values are related to the soliton order. In Eq. (1), without the self-steepening term it transforms in the conventional NLSE. Here we will be concerned only with the fundamental soliton $(N=1)$.

The self-steepening of the pulse edge arises from an intensity-dependent group velocity and produces a temporal pulse distortion and an asymmetry in the pulse spectrum. Self-steepening can develop optical shock, understood as an extremely sharp rear edge. Early work on this subject is described in Refs. [3–8], while more recent work has been reported in Refs. [9–14]. It should be pointed out that optical shocks also reveal themselves as a result of the interplay between SPM and GVD, as recently demonstrated in optical fibers through the phenomena of optical wave breaking [5]. A characteristic of optical shocks arising due to SPM and GVD alone is that it occurs in both leading and trailing edge of the pulse, symmetrically, unlike the shock induced by self-steepening, which is asymmetric in nature. In general, the MNLSE including the self-steepening term has been analytically solved [6,12]. Recently exact analytical solutions for Eq. (1) were given in Ref. [15]. Their traveling-wave method is based on a choice for the complex amplitude $U(\zeta,\tau)$ of the wave in which its modulus and phase are dependent on the variable $\eta=\tau-a_0\zeta$. Although they found all symmetric solutions for the given boundary conditions, their results do not give the asymmetric solutions that are a natural consequence of the self-steepening term and that lead to shock formation [16–20].

In this paper we show that the ansatz method is more powerful than the traveling-wave method for the study of the MNLSE. As we shall show, the analysis of the different solutions will be made through the study of the ''potential function'' as occurs in the traveling-wave method. The difference is that in our case we can obtain the asymmetric solution as well as the symmetric one and this is done in both optical regimes (anomalous and normal). It is important to mention that our approach is suitable for obtaining some symmetric solutions (such as that of Ref. [6]) but not all (such as that of Ref. [15]). Finally, for a particular class of the solutions found in this paper, we calculate the critical distance for the shock formation.

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II. EXACT SOLUTION OF THE MODIFIED NONLINEAR SCHRÖDINGER EQUATION

Now we proceed with the analysis of Eq. (1) by separating $U(\zeta,\tau)$ into the real amplitude $V(\zeta,\tau)$ and phase $\phi(\zeta,\tau)$ according to $U=V\exp(i\phi)$. We split Eq. (1) into its real and imaginary parts, yielding

$$V\phi_\zeta+\tfrac{1}{2}\sigma(V_{\tau\tau}-V\phi_\tau^2)-V^3+sV^3\phi_\tau=0, \tag{2}$$

$$V\zeta-\tfrac{1}{2}\sigma(2V_\tau\phi_\tau+V\phi_{\tau\tau})+3sV^2V_\tau=0. \tag{3}$$

Equations (2) and (3) were solved perturbatively in Ref. [8] by a power series in the parameter s, and in Ref. [6] making suitable assumptions about the initial frequency scanning (or chirp).

SPM gives rise to an intensity-dependent phase shift while the pulse shape governed by $|U(\zeta,\tau)|^2$ remains unchanged [2]. SPM-induced spectral broadening is a consequence of the dependence of $\phi(\zeta,\tau)$. This can be understood by noting that a temporally varying phase implies that the instantaneous optical frequency differs across the pulse from its central value. Self-steepening leads to an asymmetry in the SPM-broadened spectra and of the trailing edge of the pulse that eventually creates an optical shock analogous to the development of an acoustical shock on the leading edge of a sound wave. The critical distance corresponding to the shock formation can be obtained by requiring that $[|U(\zeta,\tau)|^2]_\tau$ be infinite at the shock location. For femtosecond initial pulse width $T_0<100$ fs and peak power of the incident pulse $P_0\geqslant 1$ kW, as a result, significant self-steepening of the pulse can occur over a few-centimeter-long fiber [6].

To solve the coupled pair of Eqs. (2) and (3) we make the ansatz

$$\phi_\tau=a_0+a_2V^2, \tag{4}$$

where a_0 and a_2 are constants that will be determined later. With the ansatz (4) we decouple the pair of equations (2) and (3) that now can be solved exactly. First we make some simple manipulations in order to write Eq. (3) in the form

$$(V^2)_\zeta+[(\tfrac{3}{2}sV^2-\sigma\phi_\tau)V^2]_\tau=0. \tag{5}$$

Second we substitute Eq. (4) in Eq. (5) to find

$$(V^2)_\zeta+[-\sigma a_0+(-2\sigma a_2+3s)V^2](V^2)_\tau=0, \tag{6}$$

where we used the property $(V^4)_\tau=2V^2(V^2)_\tau$. From Eq. (6) obtain directly the general solution for V as

$$V(\zeta,\tau)=f\{\tau-[-\sigma a_0+(-2\sigma a_2+3s)V^2]\zeta\}, \tag{7}$$

where f is an arbitrary function determined by the initial form of the pulse envelope. As we can see from Eq. (4) the frequency scanning of our solution is nonlinearly modulated during the pulse propagation, as opposed to the ordinary soliton solution where it is zero.

In order to integrate Eq. (2) we need now calculate ϕ_τ. This can be done from Eqs. (4) and (7) yielding for $\phi(\zeta,\tau)$

$$\phi(\zeta,\tau)=\phi_0(\zeta)+a_0\tau+a_2\int^\eta f^2(\eta')d\eta' + \tfrac{1}{2}a_2(-2\sigma a_2+3s)\zeta f^4(\eta), \tag{8}$$

where $\phi_0(\zeta)$ is an integration constant and η is given by [see Eq. (7)]

$$\eta(\zeta,\tau)=\tau-[-\sigma a_0+(-2\sigma a_2+3s)V^2(\zeta,\tau)]\zeta. \tag{9}$$

Finally we have for ϕ_ζ from Eqs. (8) and (9)

$$\phi_\zeta=k+a_2[\sigma a_0-(\tfrac{3}{2}s-\sigma a_2)V^2]V^2, \tag{10}$$

where we took for convenience $\phi_0'(\zeta)=k$, with k constant.

We insert Eqs. (4), (7), and (10) in Eq. (2) for we obtain the initial form $f(\tau)$ making $\zeta=0$ to get the second-order equation for $V(\zeta=0,\tau)=f(\tau)$,

$$V_{\tau\tau}-\sigma a_2(s-\sigma a_2)V^5-2\sigma(1-sa_0)V^3+\sigma(2k-\sigma a_0^2)V=0. \tag{11}$$

Equation (11) can be integrated once and put into a form analogous to the equation of motion of a particle in a one-dimensional potential field

$$\tfrac{1}{2}(V_\tau)^2+\Pi(V)=0, \tag{12}$$

where the potential field $\Pi(V)$ is given by

$$\Pi(V)=-\tfrac{1}{6}\sigma a_2(s-\sigma a_2)V^6 -\tfrac{1}{2}\sigma(1-sa_0)V^4+\tfrac{1}{2}\sigma(2k-\sigma a_0^2)V^2+\delta.$$

Here δ is an integration constant, and $V^2=W$. Equation (12) can be rewritten as

$$W_\tau^2+\Pi(W)=0, \tag{13}$$

where

$$\Pi(W)=W(\alpha W^3+\beta W^2+\gamma W+\delta),$$

with $\alpha\equiv-\tfrac{4}{3}\sigma a_2(s-\sigma a_2)$, $\beta\equiv-4\sigma(1-sa_0)$, and $\gamma\equiv4\sigma(2k-\sigma a_0^2)$.

III. RESULTS

In the following we show how to obtain the general solution of Eq. (13) and give some possible solutions for $\delta\neq0$. For different kinds of root distributions of the polynomial $\Pi(W)$, there are different kinds of solutions for Eq. (13). Because $W=V^2>0$ and all of the coefficients in $\Pi(W)$ are real, we will discuss the real solution of Eq. (13) only.

(1) $\alpha>0$.

(a) All four roots of the polynomial $\Pi(W)$ are real.

(i) There is a single root $W=0$ and a triple root $W=a$. In this case ''potential function'' $\Pi(W)$ can be rewritten as

$$\Pi(W)=\alpha W(W-a)^3. \tag{14}$$

$\Pi(W)<0$, when $0<W<a$. So there is a real solution for Eq. (13):

$$W=\frac{2\alpha a^3(\tau-\tau_0)^2}{1+2\alpha a^2(\tau-\tau_0)^2}, \tag{15}$$

where τ_0 is an integration constant. This is the ''algebraic'' dark soliton solution. We have the conditions $a=-\beta/3\alpha$, $3\alpha\gamma=\beta^2$, and $\delta=\beta^3/27\alpha^2$.

(ii) There are two single roots $W=0$ and b and double root $W=a$.

In this case ''potential function'' $\Pi(W)$ can be written as

$$\Pi(W)=\alpha W(W-a)^2(W-b), \tag{16}$$

when $0<W<b$ and $a>b>0$. The real solution in Eq. (13):

$$W=\frac{b\left[1-\frac{2a-b}{2(a-b)}\tan^2\phi\right]\pm b\left(\left[1-\frac{2a-b}{2(a-b)}\right]^2+\frac{a}{a-b}\left[1-\frac{(2a-b)^2}{4a(a-b)}\tan^2\phi\right]\tan^2\phi\right)^{1/2}}{2\left[1-\frac{(2a-b)^2}{4a(a-b)}\tan^2\phi\right]}, \tag{17}$$

where $\phi=\sqrt{\alpha a(a-b)}(\tau-\tau_0)$. We have the conditions

$$3a^2+2\frac{\beta}{\alpha}a+\frac{\gamma}{\alpha}=0,$$

$$b=-2a-\frac{\beta}{\alpha},$$

and

$$\delta=\alpha a^2\left(2a-\frac{\beta}{\alpha}\right).$$

(iii) There is one single root $W=0$ and three single roots $W=a$, b, and c.

In this case the ''potential function'' $\Pi(W)$ can be written as

$$\Pi(W)=\alpha W(W-a)(W-b)(W-c), \tag{18}$$

when $a>b>c>0$.

For $c>W>0$ we have the solution

$$W=\frac{ac\,\mathrm{sn}^2(\sqrt{2\alpha(a-c)b}(\tau-\tau_0),\kappa)}{a+c[\mathrm{sn}^2(\sqrt{2\alpha(a-c)b}(\tau-\tau_0),\kappa)-1]}, \tag{19}$$

where $a=1$ and b, c, κ are given by

$$b,c=(-,+)\,\frac{1}{2}\left(1+\frac{\beta}{\alpha}\right)\pm\left[\frac{1}{4}\left(1+\frac{\beta}{\alpha}\right)^2+\frac{\delta}{\alpha}\right]^{1/2},$$

$$\kappa=\left(\frac{(a-b)c}{(a-c)b}\right)^{1/2},$$

respectively, and sn is the Jacobian elliptic function. For $a>W>b$ we have the solution

$$W=\frac{(a-b)c\;\;\mathrm{sn}^2(\sqrt{2\alpha(a-c)b}(\tau-\tau_0),\kappa)}{(a-b)\mathrm{sn}^2(\sqrt{2\alpha(a-c)b}(\tau-\tau_0),\kappa)-(a-c)}, \tag{20}$$

where a, b, c, and κ are the same as above.

(b) In the case when there are two real roots $W=0$, a and a pair of conjugate complex roots $W=\pm ib$.

(i) In this case ''potential function'' $\Pi(W)$ can be written as

$$\Pi(W)=W(W-a)(W^2+b^2). \tag{21}$$

When $0\leqslant W\leqslant a$, there is a real solution for Eq. (13),

$$W=\frac{ab[1-\mathrm{cn}(\sqrt{2\alpha(a^2+b^2)}b(\tau-\tau_0),\kappa)]}{b[1-\mathrm{cn}(\sqrt{2\alpha(a^2+b^2)}b(\tau-\tau_0),\kappa)]+\sqrt{a^2+b^2}[1+\mathrm{cn}(\sqrt{2\alpha(a^2+b^2)}b(\tau-\tau_0),\kappa)]}. \tag{22}$$

(2) $\alpha<0$.

(a) All four roots of the polynomial $\Pi(W)$ are real.

(i) There is a single root $W=0$ and a triple root $W=a$. In this case, it is impossible for $\Pi(W)<0$ so there is no real solution for Eq. (13).

(ii) There are two single roots $W=0$ and b and double root $W=a$. In this case ''potential function'' $\Pi(W)$ can be written as

$$\Pi(W)=W(W-a)^2(W-b). \tag{23}$$

When $0\leqslant b<W<a$, there is a real solution for Eq. (13),

$$\ln\left[\frac{a^2b-2a^2W+2aW^2-bW^2-2\sqrt{a(a-b)}\sqrt{W(W-a)^2(W-b)}}{(W-a)^2}\right]=\sqrt{-\alpha a(a-b)}(\tau-\tau_0), \tag{24}$$

for simplicity, we let $b=0$. We have

$$W=\frac{a}{2}\,\mathrm{sech}\left[\frac{a}{2}\sqrt{-\alpha}(\tau-\tau_0)\right]\exp\left[\pm\frac{a}{2}\sqrt{-\alpha}(\tau-\tau_0)\right]. \tag{25}$$

(iii) There are four single roots $W=0$, a, b, and c. In this case, it is impossible for $\Pi(W)$. So there is no real solution for Eq. (13).

(b) In the case when there are two real roots $W=0$, a, and a pair of conjugate complex roots $W=\pm ib$. In this case, it is impossible for $\Pi(W)<0$. So there is no real solution for Eq. (13). It is important to remember that for all the solutions above we should replace τ by $\eta(\zeta,\tau)$ of Eq. (9) in order to find the amplitude $V(\zeta,\tau)$ of Eq. (7).

In all the discussion above we considered $\delta\neq 0$. As we know, the boundary conditions in which V and V_τ vanish as $\tau\rightarrow\pm\infty$ are very important in the sense of temporally localized solutions and this means here $\delta=0$. We are now going to show that in this case we can find an analytical solution that is a generalization of an early result in the literature [6] in which we depict its asymmetric nature due to the self-steepening term. Later we use this solution to show the shock formation and calculate the critical distance in which the shock occurs.

Without loss of generality we can assume again that the peak of the pulse is located at $\tau=0$, i.e., $V(0)=V_0$ and $V_\tau=0$. We also have $\Pi(V_0)=0$, which specifies the constant k as follows:

$$k=\tfrac{1}{2}\sigma a_0^2+\tfrac{1}{2}(1-sa_0)V_0^2+\tfrac{1}{6}a_2(s-\sigma a_2)V_0^4. \tag{26}$$

For the NLSE the value assumed by k is $\frac{1}{2}$ obtained by the inverse scattering method and in this case the frequency scanning is zero. We recover this result in Eq. (26) as $\phi_\tau=0$ $(a_0=0,\ a_2=0)$ and also $s=0$ (in this case MNLSE goes into the NLSE) where we made $V_0=1$ for the normalized pulse.

For a potential well to exist between $V=0$ and $V=V_0$ the coefficient of V^2 in Eq. (12) must be negative, i.e.,

$$\sigma(2k-\sigma a_0^2)<0. \tag{27}$$

The formal solution of Eq. (12) is found after some algebraic manipulations as

$$V^2(\zeta,\tau)=\frac{V_0^2}{2-v}\left[\cosh^2(\mu\eta)+\frac{v-1}{2-v}\right]^{-1}, \tag{28}$$

where

$$\mu^2=-\sigma(1-sa_0)V_0^2-\tfrac{1}{3}\sigma a_2(s-\sigma a_2)V_0^4,$$
$$v=-\frac{\sigma}{\mu^2}(1-sa_0)V_0^2, \tag{29}$$

and η is defined by Eq. (9). The results obtained in Eqs. (7), (28), and (29) are the ones from which we will make a detailed analysis showing under which conditions we can retrieve earlier results of the literature, as well as new ones. We can see from Eqs. (9) and (28) that in the anomalous regime $(\sigma=-1)$ for $a_2=-3s/2$ we recover the results of Ref. [6]. We can still emphasize some features of the obtained solution described by Eq. (28). For $\nu=0$, for example, we have from Eq. (16) that $a_0=1/s$ and the amplitude in this case goes as a square root of the conventional sech-type solution, whereas for $v=1$ $(a_2=0$ or $a_2=\sigma s)$ the amplitude has the usual sech-type behavior. This means that in going from $v=1$ to $v=0$ the input pulse width which satisfies Eq. (28) is shorter than the conventional solution. For the conventional nonlinear Schrödinger equation ''bright'' and ''dark'' solitons exist depending on whether $\sigma=\mp 1$, respectively. In our solution the condition for the existence of a bright soliton in the normal regime $(\sigma=+1)$ becomes

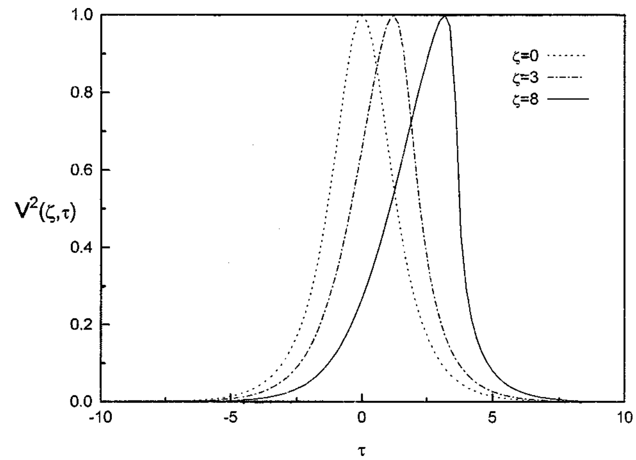

FIG. 1. Plot of normalized intensity versus time from Eq. (28) with $a_0=0.2$, $s=0.2$, and $a_2=-0.2$. ζ is the position along the fiber, showing the shift, the asymmetry, and the self-steepening of the pulse. The quantities plotted are dimensionless.

$$1+\frac{a_2}{3}(s-a_2)<sa_0, \tag{30}$$

where use of Eqs. (26) and (27) was made. This implies that in a medium in the normal regime we can still propagate a bright soliton provided Eq. (30) in the anomalous regime $(\sigma=-1)$ is

$$1+\frac{a_2}{3}(s+a_2)>sa_0. \tag{31}$$

The conditions set by Eqs. (30) and (31) will be important in the determination of the possible values of the parameters a_0. The phase is easily found from Eqs. (8) and (28) and is given by

$$\phi(\zeta,\tau)=k\zeta+a_0\tau+\frac{a_2V_0^2}{\mu\sqrt{1-\nu}}\times\tan^{-1}[\sqrt{1-\nu}\,\tanh(\mu\eta)]+\frac{a_2}{2}(3s-2\sigma a_2)\times V_0^4\zeta[(2-v)\cosh^2(\mu\eta)+v-1]^{-2}. \tag{32}$$

Once again we see from Eqs. (32) that we recover the phase of Ref. [6] when $a_2=-3s/2$ for the anomalous regime $(\sigma=-1)$. Figure 1 depicts the results of Eqs. (9), (28), and (29) where the pulse asymmetry leading to shock formation occurs, using parameters typical of current experiments [2]. The shock formation occurs because the peak of pulse moves slower than its trailing edge, but without a whole shift of the envelope.

For $a_2=0$ and arbitrary a_0 there is an asymmetry and the dispersion is unable to prevent the shock formation. In this case one should notice that, unlike the dispersionless case, the whole envelope is shifted.

IV. SHOCK FORMATION AND CRITICAL DISTANCE OF PROPAGATION

As we saw in Sec. III, the presence of the last term of Eq. (1)—the self-steepening term—is responsible for the asymmetric behavior of our solutions. This self-steepening can develop the formation of an optical shock understood as an extremely sharp rear edge seen in Fig. 1. In this section we calculate the critical distance of propagation for shock formation of the solution described by Eq. (28) with the condition that V_τ becomes infinite at the shock position. From Eq. (7) we find

$$V_\tau=\frac{(df/d\eta)}{1+(3s-2\sigma a_2)\zeta(df^2/d\eta)}. \qquad (33)$$

The condition for shock formation is then obtained from Eq. (33) and the critical distance $\zeta_{\rm cr}$ is determined when $df^2/d\eta$ is a maximum, i.e.,

$$\zeta_{\rm cr}=-\frac{1}{3s-2\sigma a_2}\frac{1}{(df^2/d\eta)_{\rm max}}. \qquad (34)$$

The maximum of $df^2/d\eta$ is calculated from Eq. (28) yielding for $\zeta_{\rm cr}$

$$\zeta_{\rm cr}=\frac{g(v)}{8\mu V_0^2(3s-2\sigma a_2)}, \qquad (35)$$

where $g(v)$ is given by

$$g(v)=\frac{[3v+\sqrt{9v^2-32v+32}]^2}{[2(3v^2-8v+8)+2v\sqrt{9v^2-32v+32}]^{1/2}}. \qquad (36)$$

As we saw in Sec. III, for $a_2=-3s/2$ in the anomalous regime $(\sigma=-1)$, we recover the results of Ref. [6] in which the pulse propagates symmetrically and consequently there is no shock formation. This is clearly corroborated by our Eq. (35) where $\zeta_{\rm cr}\to\infty$ (no shock formation) when $a_2=-3s/2$, $\sigma=-1$. From Eq. (35) since $\zeta_{\rm cr}>0$ and all other parameters involved are positive we have the conditions for a_2:$a_2\geqslant -3s/2$ $(\sigma=-1)$ and $a_2\leqslant 3s/2$ $(\sigma=+1)$, and this set a range for a_2 depending on the self-steepening parameter s. Therefore we conclude that the conditions of Eqs. (30), (31) and $a_2\geqslant 3s/2$, $a_2\leqslant -3s/2$, respectively, set a region of possible values of the parameters a_0 and a_2 that are compatible with our solution. Finally, as we said before, the self-steepening term creates an optical shock on the trailing edge of the pulse. This is due to the intensity dependence of the group velocity that results in the peak of the pulse moving slower than the wings $(\sigma=-1)$. As a result this manifests, besides the asymmetric behavior, through a shift of the pulse center. This shift can be described by the delay time $\tau_d(\zeta)$ that can be calculated from Eq. (7) making $\eta=0$ and taking $V^2(\zeta,\tau_d)=V_0^2$ as the peak of the pulse. We have then for $\tau_d(\zeta)$

$$\tau_d(\zeta)=[-\sigma a_0+(3s-2\sigma a_2)V_0^2]\zeta. \qquad (37)$$

For $\sigma=-1$, $a_2=-3s/2$, and $a_0=s$ we recover the numerical result of Ref. [6] in which the delay time has the behavior $\tau_d(\zeta)=s\zeta$ for $s<0.3$. From our result described by Eq. (37), we observe that the peak does not move $(\tau_d=0)$ for $a_0+2a_2=-3s$ with $V_0=1$.

Concerning the results presented in this paper it is important to mention that the main difference between our approach and that of Ref. [15] is related to the choice of the variable η, i.e., in their case, $\eta=\tau-a_0\zeta$ while ours is given by Eq. (9).

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